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A METHOD OF SOLVING THE INTEGRAL EQUATION OF PLANE CONTACT PROBLEMS FOR SEMI-BOUNDED BODIES†

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A typical integral equation, which arises when solving linear plane contact problems for semi-bounded bodies, is considered. By using a special representation of the kernel of this equation, an approximate method is developed for solving it that is effective over a wide range of variation of the dimensionless geometrical parameter occurring in the kernel. The method is tested on the problem of the symmetrical compression of an elastic strip along its boundaries by two similar punches.

1. THE INTEGRAL EQUATION OF PLANE CONTACT PROBLEMS

A wide range of plane contact problems for linearly deformed semi-bounded bodies can be reduced to solving the following integral equation of the first kind with a difference kernel [1–3]

$$\int_{-1}^{1} \varphi(\xi) K\left(\frac{\xi - x}{\lambda}\right) d\xi = \pi f(x), \quad |x| \leq 1, \quad \lambda \in (0, \infty) \tag{1.1}$$

$$K(y) = \int_0^\infty \frac{L(u)}{u} \cos uy\, du$$

where $\varphi(x)$ is the dimensionless contact pressure, $f(x)$ is the dimensionless indentation of the punch and λ is a dimensionless geometrical parameter. The function $L(u)$, which occurs in the expression for the kernel $K(y)$, possesses the following properties: (1) it is continuous, odd and does not vanish for any $0 \leq u < \infty$, (2) the following asymptotic expressions hold

$$L(u) = 1 + O(u^{-2}), \quad u \to \infty; \quad L(u) = Au + O(u^3), \quad u \to 0, \quad A = \text{const} \tag{1.2}$$

The following limitation is imposed on the function $f(x)$: its first derivative when $|x| \leq 1$ must satisfy the Hölder condition.

Asymptotic "large λ" ($\lambda \geq 2$) and "small λ" ($\lambda \leq 2$) methods and also the method of orthogonal polynomials and the collocation method with respect to Chebyshev nodes, also effective only for large or small λ, were developed previously in [1–3] for the approximate solution of integral equation (1.1). A method was proposed in [2, Chapter 5, Section 2, Paragraph 2] which is equally effective both of large λ and small λ, but is extremely cumbersome. Below we propose another, simpler method, equally effective, at least, for values of the parameter $\lambda \in [6, 1/3]$, i.e. over the whole range of variation of λ of practical importance.

By virtue of conditions (1.2), we can represent the function $L(u)$ in the form

$$L(u) = \text{th} Au + g(u) \tag{1.3}$$

$$g(u) = O(u^{-2}), \quad u \to \infty; \quad g(u) = O(u^3), \quad u \to 0$$

$$|g(u)| \leq \delta, \quad 0 \leq u < \infty$$

where the value of δ in practical problems is, as a rule, small compared with unity.

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Using representation (1.3) and the integral [4, (4.116(2))]

$$\int_0^\infty \frac{\operatorname{th} Au}{u} \cos uy\, du = -\ln\left|\operatorname{th}\frac{\pi y}{4A}\right|$$

we can convert integral equation (1.1) to the form

$$-\int_{-1}^{1} \varphi(\xi)\ln\left|\operatorname{th}\frac{\mu(\xi-x)}{2}\right| d\xi = \pi f(x) - \int_{-1}^{1} \varphi(\xi) F\left(\frac{\xi-x}{\lambda}\right) d\xi \tag{1.4}$$

$$F(y) = \int_0^\infty \frac{g(u)}{u}\cos uy\, du, \quad \mu = \frac{\pi}{2A\lambda}$$

We differentiate integral equation (1.4) with respect to x and obtain

$$\mu \int_{-1}^{1} \varphi(\xi)\frac{d\xi}{\operatorname{sh}[\mu(\xi-x)]} = \pi f'(x) - \frac{1}{\lambda}\int_{-1}^{1} \varphi(\xi) G\left(\frac{\xi-x}{\lambda}\right) d\xi \tag{1.5}$$

$$G(y) = -F'(y) = \int_0^\infty g(u)\sin uy\, du$$

By virtue of properties (1.3) of the function $g(u)$ it can be shown that the function $G(y)$ satisfies the Hölder condition when $|y| \leqslant R, R < \infty$. In addition, we note that the solution of the singular integral equation of the first kind (1.5) with the additional condition

$$-\int_{-1}^{1} \varphi(\xi)\ln\left|\operatorname{th}\frac{\mu\xi}{2}\right| d\xi = \pi f(0) - \int_{-1}^{1} \varphi(\xi) F\left(\frac{\xi}{\lambda}\right) d\xi \tag{1.6}$$

is equivalent to the solution of integral equation (1.4).

2. CONVERSION OF THE INTEGRAL EQUATION AND THE STRUCTURE OF THE SOLUTION

Integral equation (1.5) can be rewritten in the form

$$\mu \int_{-1}^{1} \frac{\varphi(\xi)}{\operatorname{ch}\mu\xi(\operatorname{th}\mu\xi - \operatorname{th}\mu x)} d\xi = \pi f'(x)\operatorname{ch}\mu x - \frac{\operatorname{ch}\mu x}{\lambda}\int_{-1}^{1} \varphi(\xi) G\left(\frac{\xi-x}{\lambda}\right) d\xi \tag{2.1}$$

We now make the following replacements in integral equation (2.1) and condition (1.6)

$$t = \operatorname{th}\mu x, \quad \tau = \operatorname{th}\mu\xi, \quad \psi(\tau) = \varphi(\xi)\operatorname{ch}\mu\xi, \quad h(t) = f'(x)\operatorname{ch}\mu x \tag{2.2}$$

As a result we obtain

$$\int_{-a}^{a} \frac{\psi(\tau)}{\tau - t} d\tau = \pi h(t) - \int_{-a}^{a} \psi(\tau) H(\tau, t) d\tau, \quad |t| \leqslant a \tag{2.3}$$

$$a = \operatorname{th}\mu, \quad H(\tau, t) = \frac{2A}{\pi\sqrt{(1-t^2)(1-\tau^2)}} G\left[\frac{A}{\pi}\ln\frac{(1+\tau)(1-t)}{(1-\tau)(1+t)}\right]$$

$$-\frac{1}{2}\int_{-a}^{a} \frac{\psi(\tau)}{\sqrt{1-\tau^2}}\ln\frac{1-\sqrt{1-\tau^2}}{1+\sqrt{1-\tau^2}} d\tau = \pi\mu f(0) - \int_{-a}^{a} \frac{\psi(\tau)}{\sqrt{1-\tau^2}} F\left(\frac{A}{\pi}\ln\frac{1+\tau}{1-\tau}\right) d\tau \tag{2.4}$$

It is important to note that the root singularities in (2.3) and (2.4) lie outside the ranges of definition and integration, since $a < 1$.

Any well-known approximate methods [1–3, 5–8] can be used to solve singular integral equation of the first kind (2.3). Since they are all based in some way on the exact inversion of the principal singular operator on the left-hand side of integral equation (2.3), then, for small δ (see (1.3)) and, as a consequence, small

$$\max|G(y)|, \quad \max|F(y)|, \quad 0 \leq y < \infty \tag{2.5}$$

their effectiveness will be extremely high for any values of the parameter λ (in practice for $\lambda \in [6, {}^1/_3]$).

Taking into account the above-mentioned properties of the functions $f(x)$ and $G(y)$ it can be proved [2], that if a solution of Eq. (2.3) exists for a given value of λ in the class of functions

$$\int_{-a}^{a} |\psi(\tau)|^p \, d\tau < \infty, \quad 0 < p < 2 \tag{2.6}$$

then, in general, this solution can be represented in the form

$$\psi(t) = \Psi(t)(a^2 - t^2)^{-1/2} \tag{2.7}$$

where the function $\Psi(t)$ satisfies the Hölder condition when $|t| \leq a$.

Note that the function $\psi(t)$ can be found from singular integral equation (2.3), apart from the following term

$$\pi^{-1}C(a^2 - t^2)^{-1/2} \tag{2.8}$$

Hence the constant C is then determined from additional condition (2.4).

In a number of problems of the function $f(x)$ in (1.1) can only be determined apart from the linear part $c_0 + c_1x$. To obtain c_0 and c_1 additional conditions are needed, for which the following are usually used

$$\varphi(\pm 1) = 0 \tag{2.9}$$

3. THE EXACT SOLUTION ION A SPECIAL CASE

If the function $g(u) \equiv 0$ in expression (1.3), then Eq. (2.3) degenerates ($H(\tau, t) \equiv 0$) into the classical singular integral equation of the first kind with a Cauchy kernel, solvable in closed form (see, for example, [2]). Reverting, in this solution, to the old variables and notation given by (2.2), we have

$$\varphi(x) = \frac{1}{\pi X(x)}\left[C - \mu \int_{-1}^{1} \frac{f'(\xi)Y(\xi)}{\operatorname{th}\mu\xi - \operatorname{th}\mu x} d\xi\right] \tag{3.1}$$

$$X(x) = \operatorname{ch}\mu x\sqrt{a^2 - \operatorname{th}^2\mu x}, \quad Y(x) = (\operatorname{ch}\mu x)^{-1}\sqrt{a^2 - \operatorname{th}^2\mu x}$$

where the constant C must be determined from the condition

$$-\int_{-1}^{1} \varphi(\xi)\ln\left|\operatorname{th}\frac{\mu\xi}{2}\right| d\xi = \pi f(0) \tag{3.2}$$

For conditions (2.9) we have

$$\varphi(x) = -\frac{\mu}{\pi} Y(x) \int_{-1}^{1} \frac{f'(\xi)}{(\operatorname{th}\mu\xi - \operatorname{th}\mu x)X(\xi)} d\xi \tag{3.3}$$

while conditions (2.9) themselves take the form

$$C + \mu \int_{-1}^{1} \frac{f'(\xi)\operatorname{th}\mu\xi}{X(\xi)} d\xi = 0, \quad \int_{-1}^{1} \frac{f'(\xi)}{X(\xi)} d\xi = 0 \tag{3.4}$$

Condition (3.2) also remains in force here.

In the special case when $f(x) \equiv f = \text{const}$, we obtain from formulae (3.1) and (3.2)

$$\varphi(x) = \frac{\mu f}{\mathbf{K}(\sqrt{1-a^2})X(x)}, \quad P = \frac{2f\mathbf{K}(a)}{\mathbf{K}(\sqrt{1-a^2})}; \quad P = \int_{-1}^{1} \varphi(\xi)d\xi \tag{3.5}$$

where P is the integral characteristics, defined by the last formula of (3.5) and $\mathbf{K}(a)$ is the complete elliptic integral of the first kind. In deriving formulae (3.5) we used the integral [9, (2.6.16(18))]

$$\int_{-1}^{1} \ln\frac{1+\sqrt{1-a^2x^2}}{1-\sqrt{1-a^2x^2}} \frac{dx}{\sqrt{(1-x^2)(1-a^2x^2)}} = 2\pi\mathbf{K}(\sqrt{1-a^2}) \tag{3.6}$$

4. THE APPROXIMATE SOLUTION IN THE GENERAL CASE

For the approximate solution of integral equation (2.3) with condition (2.4), when the function $g(u)$ in expression (1.3) is not identically equal to zero, it is best to use the Multopp–Kalandiya method [2, 3, 7]. We will describe in briefly here in a somewhat non-traditional form.

We substitute expression (2.7) into relations (2.3) and (2.4) and change to the new variables

$$\tau = a\cos\omega, \quad t = a\cos\theta \tag{4.1}$$

As a result we obtain

$$\frac{1}{a}\int_0^\pi \frac{\Omega(\omega)d\omega}{\cos\omega - \cos\theta} d\omega = \pi k(\theta) - \int_0^\pi \Omega(\omega)H(a\cos\omega, a\cos\theta)d\omega, \quad 0 \leqslant \theta \leqslant \pi \tag{4.2}$$

$$-\int_0^\pi \Omega(\omega)\ln|a\cos\omega|\,d\omega = \pi\mu f(0) - \int_0^\pi \Omega(\omega)M(a\cos\omega)d\omega \tag{4.3}$$

$$M(\tau) = \frac{1}{\sqrt{1-\tau^2}}\left[(\sqrt{1-\tau^2}-1)\ln|\tau| + \ln(1+\sqrt{1-\tau^2}) + F\left(\frac{A}{\pi}\ln\frac{1+\tau}{1-\tau}\right)\right]$$

where $\Omega(\theta) = \Psi(a\cos\theta)$, $k(\theta) = h(a\cos\theta)$.

The function $\Psi(t)$ we construct a Lagrange interpolation polynomial at the nodes

$$t_n = a\cos\theta_n, \quad \theta_n = \pi(2n-1)/(2N), \quad n = 1, 2, \ldots, N \tag{4.4}$$

which are the zeros of the Chebyshev polynomial of the first kind $T_N(t/a)$. In special cases when $\Psi(t)$ is an even or odd function and $N = 2r + 1$ $(r \geqslant 1)$, these polynomials have the following form respectively [3]

$$\Omega(\theta) \approx \frac{1}{r+\frac{1}{2}}\sum_{n=1}^{r+1} \Omega(\theta_n)\delta_n\left(1 + 2\sum_{m=1}^{r} \cos 2m\theta_n \cos 2m\theta\right) \quad n = 1, 2, \ldots, r+1, \tag{4.5}$$

$$\Omega(\theta) \approx \frac{2}{r+\frac{1}{2}}\sum_{n=1}^{r} \Omega(\theta_n)\sum_{m=1}^{r} \cos(2m-1)\theta_n \cos(2m-1)\theta, \quad n = 1, 2, \ldots, r$$

where $\delta_n = 1$ $(n \neq r+1)$, $\delta_n = \frac{1}{2}$ $(n \neq r+1)$.

Substituting the approximate expression for $\Omega(\theta)$ in one of the forms in (4.5) into Eq. (4.2) and using the relation [4, (7.344(1))]

$$\int_0^\pi \frac{\cos l\omega}{\cos\omega - \cos\theta} d\omega = \pi\frac{\sin l\theta}{\sin\theta}, \quad 0 \leqslant \theta \leqslant \pi, \quad l = 0, 1, \ldots \tag{4.6}$$

we can evaluate the integral on the left-hand side of Eq. (4.2) exactly. To evaluate the integral on the right-hand side of the equation approximately we use the Gauss quadrature formula [3, 7]

$$\int_0^\pi f(\omega)d\omega = \frac{\pi}{N}\sum_{n=1}^{N} f(\theta_n) \tag{4.7}$$

After evaluating the integrals in (4.2) we put $\theta = \theta_s$ in the relation obtained and thereby arrive at a system or r linear algebraic equations in the quantities

$$\sum_{n=1}^{r+1-\varkappa}\Omega(\theta_n)\delta_n\left\{\frac{1}{a\sin\theta_s}\chi_r^{(\varkappa)}(\theta_n,\theta_s)+\frac{1}{2}[H(a\cos\theta_n,a\cos\theta_s)-\right.$$

$$\left.-(-1)^{\varkappa}H(a\cos\theta_n,-a\cos\theta_s)]\right\}=\left(r+\frac{1}{2}\right)k(\theta_s),\quad s=1,2,\ldots,r \tag{4.8}$$

$$\chi_r^{(\varkappa)}(\omega,\theta)=2\sum_{m=1}^{r}\cos(2m-\varkappa)\omega\sin(2m-\varkappa)\theta$$

where $\varkappa = 0$, for the even version and $\varkappa = 1$ for the odd version.

To close the system of equations (4.8) for the even version we substitute $\Omega(\theta)$ in the first form of (4.5) into the addition condition (4.3). Using the relation [2.8]

$$-\int_0^{\pi}\cos 2m\omega\ln|a\cos\omega|\,d\omega=\begin{cases}\pi\ln(2/a), & m=0\\ \pi(-1)^m/(2m), & m\neq 0\end{cases}$$

we can accurately evaluate the integral on the left-hand side of relation (4.3). To evaluate the integral on the left-hand side of relation (4.3) approximately we again use the quadrature formula (4.7). We thereby obtain one more equation

$$\sum_{n=1}^{r+1}\Omega(\theta_n)\delta_n\left[\ln\frac{2}{a}+\sum_{m=1}^{r}(-1)^m\frac{\cos 2m\theta_n}{m}+M(a\cos\theta_n)\right]=\left(r+\frac{1}{2}\right)\mu f(0) \tag{4.9}$$

which supplements system (4.8) for the even version.

After solving system (4.8), (4.9) for the even version and system (4.8) for the odd version for $\Omega(\theta)$, from formulae (4.5) we can obtain approximate expressions for the functions $\Omega(\theta)$ and, consequently, also for the function $\Psi(t)\to\psi(t)\to\varphi(x)$.

We will again obtain a formula for evaluating the integral characteristic, defined by the last formula of (3.5). We carry out the sequence of transitions $\varphi(\xi)\to\psi(t)\to\Psi(t)\to\Omega(\omega)$ in the integrand in this formula and then substitute into it $\Omega(\theta)$ in the form defined by the first formula of (4.5). We obtain

$$P=\frac{2}{\mu(r+1/2)}\sum_{n=1}^{r+1}\Omega(\theta_n)\delta_n\left[J_0+2\sum_{m=1}^{r}(-1)^m\cos 2m\theta_n J_{2m}\right] \tag{4.10}$$

$$J_{2m}=\int_0^{\pi/2}\frac{\cos 2m\omega}{\sqrt{1-a^2\sin^2\omega}}d\omega$$

All the integrals J_{2m} can be expressed in terms of complete elliptic integrals of the first kind $\mathbf{K}(a)$ and the second kind $\mathbf{E}(a)$. We will give the formulae for the first four integrals:

$$J_0=\mathbf{K}(a),\quad J_2=[(-2+a^2)\mathbf{K}(a)+2\mathbf{E}(a)]a^{-2}$$

$$J_4=[(16-16a^2+3a^4)\mathbf{K}(a)+(-16+8a^2)\mathbf{E}(a)](3a^4)^{-1} \tag{4.11}$$

$$J_6=[(-256+384a^2-158a^4+15a^6)\mathbf{K}(a)+(256-256a^2+46a^4)\mathbf{E}(a)](15a^6)^{-1}$$

5. EXAMPLE

We will consider, as an example, the well-known problem [1] of the compression of an elastic strip ($|x|<\infty$, $-h\leqslant y\leqslant h$) along the boundaries by two similar punches, so that there is symmetry about the $y=0$ axis. Assuming that there are are no friction forces along the lines of contact of the punches with the strip, this problem can be reduced to solving integral equation (1.1) in which the function $\varphi(x)$ represents the dimensionless contact pressure, the function $f(x)$ is related to the indentation of the

Table 1

λ	$\Omega(\theta_n)/f$				P/f			
	$n = 1$	2	3	4	(4.10)	$\lambda \geqslant 2$	$\lambda \leqslant 2$	(3.5)
1/3	5.8731	5.5828	5.7068	5.6136	12.71			12.88
1/2	3.9154	7.7218	3.8045	3.7424	8.71		8.76	8.88
1	1.9541	1.8585	1.8912	1.8626	4.72		4.74	4.88
2	0.9338	0.9024	0.8908	0.8845	2.74	2,75	2.70	2.88
4	0.3949	0.3903	0.3860	0.3844	1.79	1.82		1.88
6	0.2301	0.2289	0.2275	0.2269	1.46			1.53

punches and the form of their base, the parameter λ is the ratio of the thickness of the strip $2h$ to the length of the line of contact, and the function $L(u)$ has the form

$$L(u) = \frac{\operatorname{ch} 2u - 1}{\operatorname{sh} 2u + 2u} \tag{5.1}$$

For dimensionless values of the forces P, which impress the punches into the strip, we will have the last formula of (3.5). In the second relation of (1.2) the quantity $A = 1/2$, and we then have $a = \operatorname{th}(\pi/\lambda)$. In expression (1.3) $\delta \approx 0.08$, while in (2.5) $\max|G(y)| \approx 0.18$ and $\max|F(y)| \approx 0.10$.

We will give the results of using the proposed method to solve the problem in the special case of a flat punch $f(x) \equiv f = \text{const}$. The calculations were carried out for $r = 3$ for 10 values of the numbers.

On the left-hand side of Table 1 we give values of $\Omega(\theta_n)/f(n = 1, 2, 3, 4)$, calculated for different λ and of the dimensionless contact pressure necessary for the calculation using the formula

$$\varphi(x) \approx \frac{1}{(r + \frac{1}{2}) \operatorname{ch} \mu x \sqrt{a^2 - \operatorname{th}^2 \mu x}} \sum_{n=1}^{r+1} \Omega(\theta_n)\delta_n \left[1 + 2\sum_{m=1}^{r} \cos 2m\theta_n T_{2m}\left(\frac{\operatorname{th} \mu x}{a}\right)\right] \tag{5.2}$$

where $T_{2m}(t)$ are Chebyshev polynomials.

On the right-hand side of Table 1 we show the results of a calculation of P/f from formula (4.10), obtained by the "large λ" method [1, Table 7] and the "small λ" method [1, Table 11] (the corresponding columns in the table are denoted by $\lambda \geqslant 2$ and $\lambda \leqslant 2$) and obtained from the second formula of (3.5) (i.e. assuming that the additional term $g(u)$ in formula (1.3) can be neglected).

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